Highlights of Calculus

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1 ϵ, δ for Limits

Danger case:

$$\begin{aligned} \infty - \infty \\ 0 \cdot \infty \\ \frac{0}{0} \\ 0^0 \text{ or } 1^\infty \end{aligned}$$

L'Hospital Rule:

$$\frac{f(x)}{g(x)} \to \frac{\frac{\Delta f}{\Delta x}}{\frac{\Delta g}{\Delta x}} \to \frac{f'}{g'}$$

For any small ϵ chosen, we can find $\delta > 0$, so that if $|f(x) - f(a)| < \epsilon$, then $|f(x) - f(a)| < \delta$

2 Fundamental Theorem of Calculus

$$
\begin{aligned}
f'(x) &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \frac{df}{dx}, \text{When} \Delta x \to 0
\end{aligned}
$$

Here, Δx means two point difference in x, df means difference in function value caused by the differenc in x。$\Delta x \to 0$ 就是代数到微积分的过程。

考虑函数值 x 在点 $x_1, x_2, ..., x_n$ 的函数值 $f_1, f_2, ..., f_n$, 进而考虑其两者之间的差值 $(f_2 - f_1) + (f_3 - f_2) + \cdots + (f_n - f_{n-1)} = f_n - f_1$。从这里可以简单的理解为，你可以将一个函数，利用其差值累加还原为原函数的值，这就是积分的过程；

$$
\begin{aligned}
\sum \Delta y &= y_{\text{last}} - y_{\text{first}} \\
f(x) &= \int f'(x) dx = \sum \frac{\Delta y}{\Delta x} \cdot \Delta x, \text{Where} f'(x) dx = df, \text{when} \Delta x \to 0
\end{aligned}
$$

从这里可以看出，对于导函数可将其视为用高度函数表示原函数的函数，其高度与其底部“面积”的乘积表示了其空间大小，即原函数的差值。

对于微分还有另一种理解为变换的视角，即从一个函数变换到另一个函数-线性映射，这个映射操作的符号记做 $\frac{d}{dx}$, 它将 y 进行变换到 y', $y' = \frac{d}{dx} \cdot y$

二阶导数的定义如下：

$$
y'' = \frac{d^2 y}{dx^2}
$$

对于这里的符号解释如下：

> 对于 dx^2, 只是对于 x 只是进行了两次除法操作即 $\frac{\Delta \Delta f}{\Delta x \cdot \Delta x}$，但是对于 y 而言则是在第一次的 df 之上再次取差值即 $d(df)$, 也就是求差值这个操作 d(diffence) 重复了两次。

$$
\begin{aligned}
f''(x) > 0 &\to \text{convex function} \\
f''(x) < 0 &\to \text{concave function}
\end{aligned}
$$

关于一阶，以及二阶导数的主要应用在于寻找各个特殊的点。

$$
\begin{aligned}
f'(x) &\to \text{stationary point} \\
f''(x) &\to \text{inflection point} \\
f'(x) = 0, \text{and} f''(x) > 0 &\to \text{Local max} \\
f'(x) = 0, \text{and} f''(x) < 0 &\to \text{Local min}
\end{aligned}
$$

对于函数的最值，则需要比较所有极值点以及边界点确定。

3 Derivatives of $e^x, \sin x, \cos x, x^n$

3.1 Exponential Function

Key: Which function's derivatives are equal to the function itself?

$$\frac{df}{dx} = y \to \text{first differential equation}$$

Construction:

$$
\begin{aligned}
y(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \dots + \frac{1}{n!}x^n + \dots \\
\frac{df}{dx} &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \dots + \frac{1}{n!}x^n + \dots
\end{aligned}
$$

这里思想在于当 when$x = 0, e^x = 1$, 那么其导数也为 1; 导数为 1，原函数为什么其导数才为 1 呢？如此反复迭代；显然当 $n \to \infty$，两式才相等。该级数称之为指数级数。

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \dots + \frac{1}{n!}x^n + \dots$$

set $x = 0$, $e = 1 + 1 + \dots = 2.71828\dots \to$ Euler's Number

用指数级数可证明指数函数下面的性质

$$e^a \cdot e^b = e^{a+b}$$

Euler's Number 也可以通过如下方式计算得到

$$e = (1 + \frac{1}{N+1})^N, \text{When} N \to \infty$$

对于该式子的展开基于二项式定理 (Binomial Theorem).

$$\frac{dy}{dx} = y$$

$$y = f(x) = 1 + x + \frac{1}{2}x^2 + ... + \frac{1}{n!}x^n + ... = e$$

3.2 Trigonometric Function

三角函数起源于勾股定理

$$a^2 + b^2 = c^2$$
$$(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$$
$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

三角函数求导关键在于用半径为 1 的圆描述周期运动，以及其中的三角形。

下面给两个重要的极限

$$\sin\theta < \theta \to \frac{\sin\theta}{\theta} < 1$$
$$\tan\theta > \theta \to \frac{\sin\theta}{\theta} > \cos\theta$$
$$\frac{\sin\theta}{\theta} = 1, \text{when} \theta \to 0$$

前两个式子可由弧度制的弧长和面积证明，该极限可认为是 $\sin 0$ 处的导数, 由上面两个式子夹逼准则定义。

下面给出另一个重要的极限。

$$\frac{\cos\theta - 1}{\theta} = 1, \text{when} \theta \to 0$$

该极限可认为是 $\cos 0$ 处的导数。

$$
\begin{aligned}
\frac{\Delta \sin x}{\Delta x} &= \frac{\sin(x+\Delta x)-\sin x}{\Delta x} \\
&= \frac{\sin x(\cos\Delta x - 1)}{\Delta x} + \frac{\sin\Delta x\cos x}{\Delta x} \\
&= \cos x
\end{aligned}
$$

仿照上例子可得到 $\cos\theta$ 的导数；下面不加证明地给出 $\cos x$ 的导数

$$\frac{d\cos x}{dx} = -\sin x$$

3.3 Product Rule, Quotient Rule, Derivaitives to Power Function

$q(x) = f(x)g(x)$

考虑边长分别为 $f(x), g(x)$, 的长方形，当两边分别改变 Δx，其面积的变化：

$$\Delta \text{area} = f(x)g(x+\Delta x) - g(x)) + g(x)(f(x+\Delta x - f(x))) + \Delta x^2$$

When $\Delta x \to 0$,

$$
\begin{aligned}
dq &= f(x)dg + g(x)df \\
\frac{dq}{dx} &= f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}
\end{aligned}
$$

Quation rule 可由乘法法则推导得到。

$$\frac{f(x)}{g(x)} = \frac{f(x)g' - g(x)f'}{g(x)^2}$$

4 Chain Rule, and Derivatives of Inverse Function $\ln x, \sin^{-1} x, \cos^{-1} x$

4.1 Chain Rule

$$f'(y(x)) = \frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}$$

对于偶函数，其导数为奇函数。对于奇函数，其导数为偶函数。

$$y = f(x) \to x = f^{-1}(y)$$

需要注意的是只有在单调区间内，才有逆函数，且 f 与 f^{-1} 的函数图像关于原点对称。

4.2 Logarithmic Function

指数函数的逆函数为对数函数，其求的是指数的值。

$$x = \ln y$$

其具有如下性质

$$\ln ab = \ln a + \ln b$$
$$\ln y^n = n \ln y$$

4.3 Derivatives for $\ln x, \sin^{-1} x, \cos^{-1} x$

set

$$y = e^x$$
$$x = \ln y$$

Then

$$y = e^x \to e^{\ln y} = y$$

$$e^{\ln y} \cdot \frac{d \ln y}{dy} = 1, \text{Where} e^{\ln y} = y$$

set

$$y = \sin x$$
$$x = \sin^{-1} y$$

Then

$$\sin \sin^{-1} y = y$$
$$\cos \sin^{-1} y \cdot \frac{d \sin^{-1} y}{y} = 1, \text{Where} \cos \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$$

Note that the $\sin^{-1} y$ is an angle.

Give the $\frac{d \cos^{-1} y}{dy}$ without proof.

$$\frac{d \cos^{-1} y}{dy} = -\frac{1}{\sqrt{1-y^2}}$$

Note that:

$$\frac{d \cos^{-1} y}{dy} + \frac{d \sin^{-1} y}{dy} = 0$$

Where $\theta + \alpha = \frac{\pi}{2}$ is a constant.

Some other deritivites:

$$\frac{d \arctan x}{x} = \frac{1}{1+x^2}$$
$$\frac{d \text{acrcot} x}{x} = -\frac{1}{1+x^2}$$
$$\frac{d a^x}{x} = a^x \ln a$$

Converion between different base.

$$\log_a |x| = \frac{1}{x \ln a}$$
$$\log_a b = \frac{\ln b}{\ln a} = \frac{\log_n b}{\log_n a}$$

5 Growth Rate and Logarithmic Plot

各函数的增长速度如下，其倒数就是减慢的速度。

$$\begin{matrix} CX\ldots & x^2, x^3\ldots 2^x, e^x, 10^x\ldots & x!x^x \\ \text{Linear} & \text{PolynomialExponential} & \text{Factorial} \end{matrix}$$

对数尺度能够处理极大或者极小（ $x \to 0$ ）的值, 但是该尺度下是没有 0 的。

对数尺度能够将非线性问题转换为线性问题

$$y = AX^n \to \log y = \log A + n \log X, \text{logarithmic plot}$$
$$y = B10^{Cx} \to \log y = \log B + Cx, \text{semi-logarithmic plot}$$

6 Linear Approximation/Newton's Method

$$f(x) = f(a) + f'(a)(x - a)$$

$$F(x) = 0 \to x - a = \frac{F(a)}{F'(a)}$$

The core of Newton's method is iteration.

7 Power Series/Euler's Great Formula

幂级数的核心在于用多项式进行函数的近似，用多项式近似的好处在于其 n 阶导数只和第 n 阶项有关，其它在此之前的多项式都为 0，第 n 阶项的系数为 $n!$。

考虑指数级数，在 0 处的 $0, 1, 2, \ldots, n$ 导数值。

$$1, 1, 1, \ldots, 1$$

为了匹配这个系数，对于幂函数的 n 阶项的导数系数 $n!$ 除 $n!$ 则可匹配每一阶的系数。

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + \ldots$$

仿照上例，给出 $\sin x, \cos x$ 的幂级数

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

对于欧拉公式，可由上面三个级数给出

$$e^{i\theta} = 1 + ix + \frac{1}{2}(ix)^2 + \frac{1}{6}(ix)^3 + \ldots$$

整理之后可见，右边即为 $\sin x, \cos x$ 的幂级数。

$$e^{i\theta} = \cos x + i \sin x$$

欧拉公式给出了在横轴为实数，纵轴为复数的复平面上，数据之间的关系。

下面给出两个其它重要的幂级数

$$\text{Geometrix series} \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \ldots, \text{Where} 0 < |x| < 1$$
$$\text{Integrated from the above equation} - \ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \text{ Where } x < 1$$

8 Differential Equations

8.1 Differential Equations of Motion

Linear, and Second order equation.

$$m\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + ky = 0$$

When $m = 0$

$$\frac{dy}{dt} = ay \to y = ce^{at}$$

When $r = 0$

$$\frac{d^2y}{dt^2} = \frac{k}{m}y = -\omega^2 y \to y = C\cos\omega t + D\sin\omega t$$

When $m = r = 0$

$$\frac{d^2y}{dt^2} = 0 \to y = C + Dt$$

General solutaion - Try $y = e^{\lambda t}$

$$m\lambda^2 + 2r\lambda + K = 0$$

Three Cases:

$$y'' + 6y' + 8y = 0 \to y(t) = Ce^{-2t} + De^{-4t}$$
$$y'' + 6y' + 10y = 0 \to y(t) = Ce^{(-3-i)t} + De^{(-3+i)t}$$
$$y'' + 6y' + 9 = 0 \to y(t) = Ce^{-3t} + Dte^{-3t}$$

8.2 Differential Equations of Growth

The growth rate proportional to itself.

$$\frac{dy}{dt} = cy$$
$$y(0) \to \text{Given start}$$
$$y(t) = y(0)e^{ct}$$

Add source term:

$$\frac{dy}{dt} = cy + s\text{Where}s\text{is source term}$$
$$\frac{d(y+\frac{s}{c})}{dt} = c(y + \frac{s}{c})$$
$$y + \frac{s}{c} = (y(0) + \frac{s}{c})e^{ct}$$

For Linear eq, the solutions to eq have form below

$$y(t) = y_{\text{particular}}(t) + y_{\text{right side 0}}(t)$$

Specially for $\frac{dy}{dt} = cy + s$

$$y_{\text{particular}} = -\frac{s}{c}$$
$$y_{\text{set s = 0}} = Ae^{ct}$$

Then

$$y = -\frac{s}{c} + Ae^{ct}$$

To find A, put $t = 0$, $y(0) = \frac{s}{c} + A$

Non-linear equation for population:

$$\frac{dp}{dt} = cp - sp^2$$

To solve this equation, set $y = \frac{1}{p}$ to turn this equation to linear equation.

Equation for predators and prey

$$\frac{du}{dt} = -cu + suv$$
$$\frac{dv}{dt} = cv - suv$$

9 Six Functions, Six Rules, and Six Theorems

Six Functions

$$\begin{aligned}
\frac{1}{n+1}x^{n+1} &\to x^n & &\to (n-1)x^{n-1} \\
-\cos x &\to \sin x & &\to \cos x \\
\sin x &\to \cos x & &\to -\sin x \\
\frac{1}{c}e^{cx} &\to e^{cx} & &\to ce^{cx} \\
x\ln x - x &\to \ln x & &\to \frac{1}{x}\text{power -1}
\end{aligned}$$

Ramp Function

Six Rules

$$af(x) + bg(x) \to a\frac{df}{dx} + b\frac{dg}{dx}$$
$$f(x)g(x) \to f(x)\frac{dg}{dx} + \frac{df}{dx}(gx)$$
$$\frac{f(x)}{g(x)} \to \frac{gf' - fg'}{g^2}$$
$$x = f^{-1}(y) \to \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$
$$f(g(x)) \to \frac{df}{dy} \cdot \frac{dy}{dx}$$
$$\text{L'Hospital}\frac{f}{g} = \frac{\frac{df}{dx}}{\frac{dg}{dx}}\text{When}x \to a, f(a), g(a) \to 0$$

Six Theorems

- Fundamental Theorem of Calculus
- Mean Values Theorem

- Taylors Series/Theorem
- Bionomial Theorem - Taylor at a = 0 $\rightarrow$ Pascal triangle

$$f(x) = (1 + x)^p = 1 + px + \frac{p(p-1)}{2 \cdot 1}x^2 + ...$$